

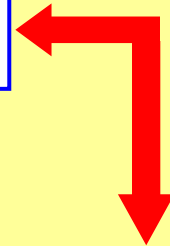
Recall from slide 2, lecture 6:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = \frac{2\pi}{\hbar} \frac{V_n}{c} \frac{1}{d\Omega} |M_{if}|^2 \rho_f$$

$$\left\{ \begin{array}{l} M_{if} = \int \psi_f^* V(\vec{r}) \psi_i d^3r \\ \rho_f = dn / dE_F \end{array} \right.$$



for the cross-section:



$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = \frac{2\pi}{\hbar} \frac{1}{c V_n} \left(\frac{Z e^2}{4\pi \epsilon_o} \right)^2 \left(\frac{4\pi}{q^2 + \alpha^{-2}} \right)^2 \left(F(q^2) \right)^2 \frac{dn}{dE_F d\Omega}$$

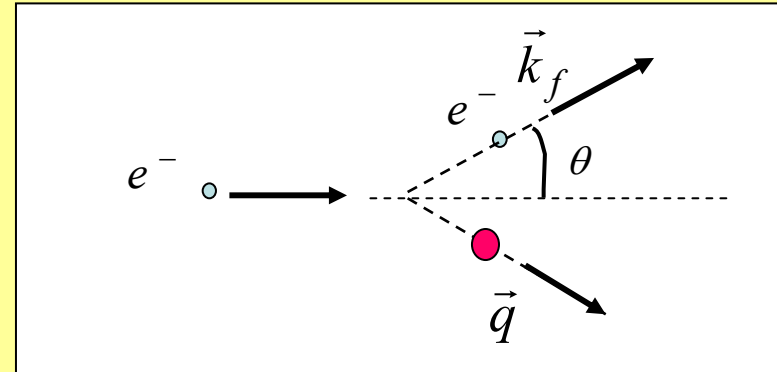
All we have left to calculate is the "density of states" factor, where E_F is the total energy in the final state when the electron scatters at angle θ , and this factor accounts for the number of ways it can do that.

$$\frac{dn}{dE_F d\Omega}$$

Consider the **total** final state energy:

2

$$\frac{dn}{dE_F d\Omega} = \frac{dn}{dp_f d\Omega} \left(\frac{dp_f}{dE_F} \right)_\theta$$



$$E_F = E' + E_R$$

(electron)

(recoil)

$$E_F = (cp_f + mc^2) + (Mc^2 + K)$$

(being careful with the factor of c !)

$$dE_F \cong c dp_f$$

$$\frac{dp_f}{dE_F} = \frac{W p_f}{Mc^3 p_i} \approx \frac{1}{c}$$

$$\frac{dn}{dE_F d\Omega} = \frac{dn}{cdp_f d\Omega}$$

This is useful because the momentum states are quantized - we have our electrons in a normalization volume, and we can "count the states" inside ...

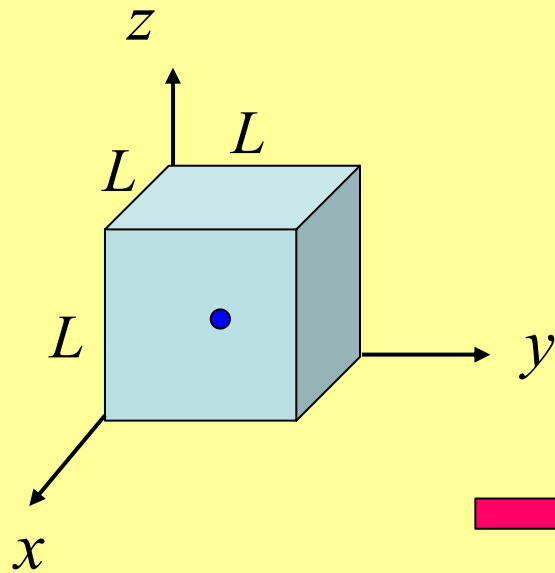
Recall the wave function:

$$\psi_f(\vec{r}) = \frac{1}{\sqrt{V_n}} e^{i\vec{k}_f \cdot \vec{r}} \quad \text{with } p_f = \hbar k_f$$

The normalization volume is arbitrary, but we have to be consistent

let $V_n = L^3$, i.e. the electron wave function is contained in a cubical box.

Use **periodic boundary conditions** - wave function is the same on all sides of the box



Since: $\vec{k}_f \cdot \vec{r} \equiv k_x x + k_y y + k_z z$

Then it follows that:

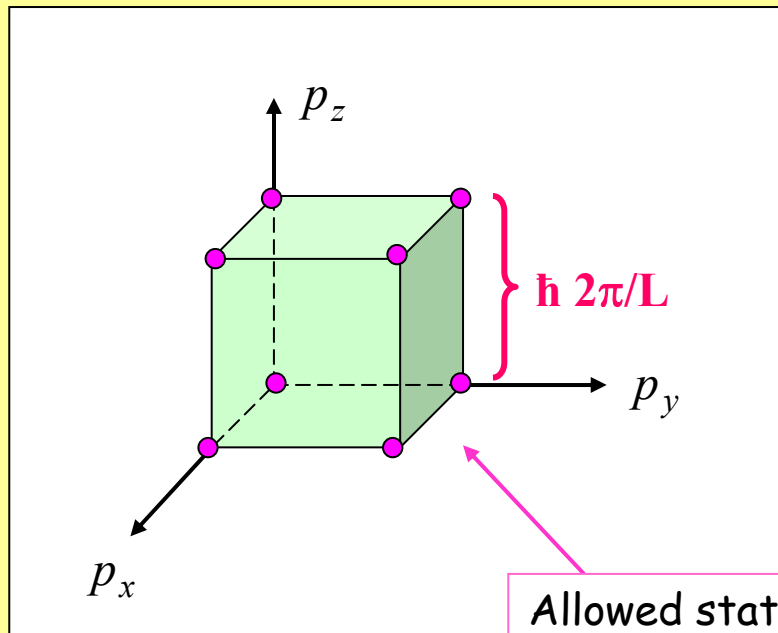
$$\psi_f(x, y, z) = \frac{1}{\sqrt{L^3}} e^{ik_x x} e^{ik_y y} e^{ik_z z}$$

$$k_x L = n_x 2\pi, \text{ etc....}$$

So, momentum is quantized on a 3-d lattice:

4

$$\vec{p}_f = \hbar \vec{k}_f = \hbar \left(\frac{2\pi}{L} \right) (n_x \hat{i} + n_y \hat{j} + n_z \hat{k})$$
$$n_x = \pm (1, 2, 3 \dots) \text{ etc.}$$



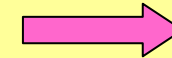
For a relativistic electron beam, the quantum numbers n_x etc. are very large, but finite.

We use the quantization relation **not** to calculate the allowed momentum, but rather to calculate **the density of states!**

Allowed states are dots, 1 per cube of volume $\tau_p = (2\pi\hbar/L)^3$

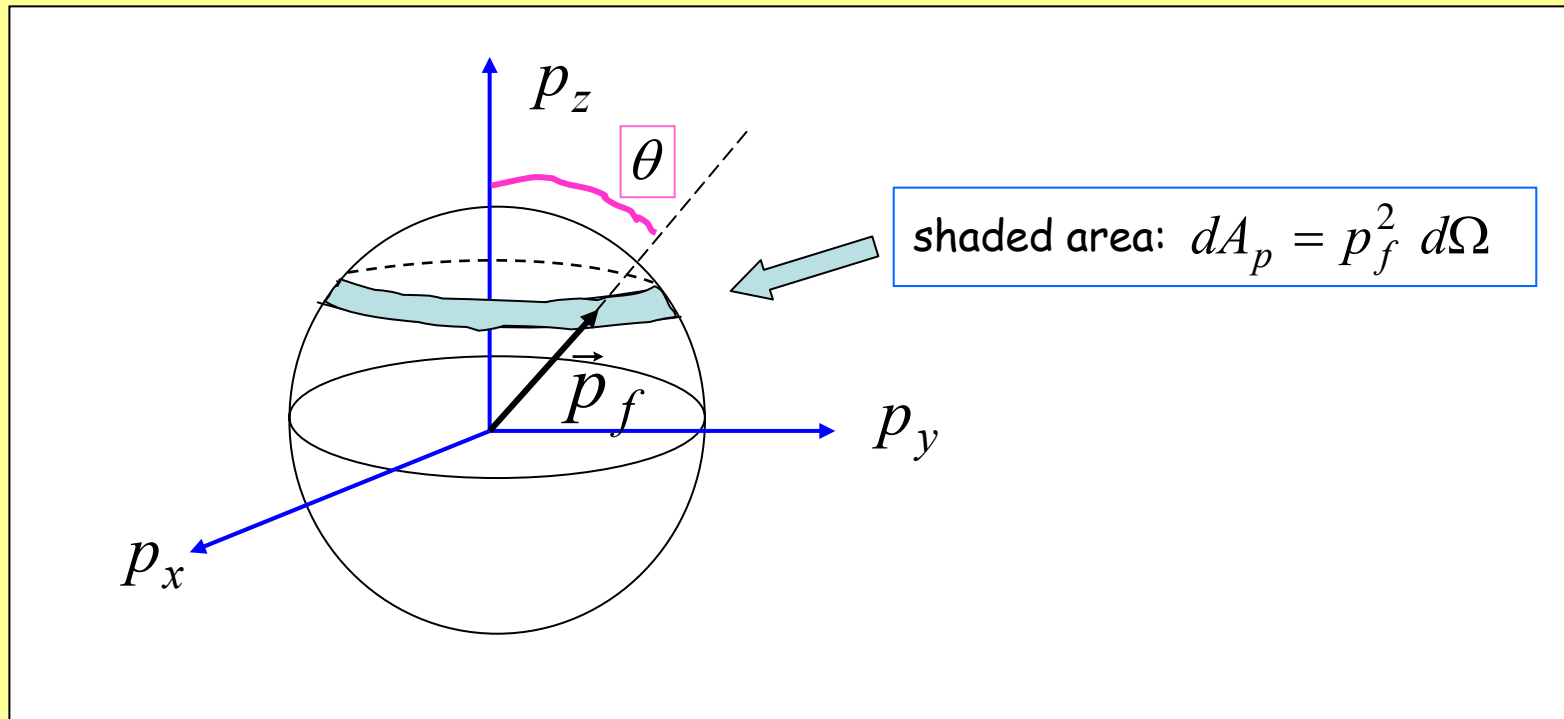
See F&H section 10.2 ...

$$\frac{dn}{d\tau_p} = \frac{1 \text{ state}}{(2\pi \hbar / L)^3}$$



Finally, consider the scattered momentum into $d\Omega$ at θ :

5



number of momentum points in the shaded ring: $dn = \left(\frac{dn}{d\tau_p} \right) \times (dA_p dp_f)$


→
$$dn = \frac{V_n}{(2\pi \hbar)^3} p_f^2 dp_f d\Omega$$

$$dn = \frac{V_n}{(2\pi \hbar)^3} p_f^2 dp_f d\Omega$$

We want the density of states factor:

$$\frac{dn}{dE_F d\Omega} = \frac{dn}{c dp_f d\Omega} = \frac{V_n}{(2\pi \hbar)^3} \frac{p_f^2}{c}$$

FINALLY, from slide 10:

$$\begin{aligned} \left(\frac{d\sigma(\theta)}{d\Omega} \right) &= \frac{2\pi}{\hbar} \frac{1}{c \cancel{V_n}} \left(\frac{Z e^2}{4\pi \epsilon_0} \right)^2 \left(\frac{4\pi}{q^2 + \alpha^{-2}} \right)^2 \left(F(q^2) \right)^2 \left(\frac{\cancel{V_n}}{(2\pi \hbar)^3} \frac{p_f^2}{c} \right) \\ &= (\text{point charge cross-section}) \times \left(F(q^2) \right)^2 \end{aligned}$$


Result: Cross section for electron scattering from nuclear charge Z:

7

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = \frac{4 Z^2}{\hbar^2 (\hbar c)^2} \left(\frac{e^2}{4\pi \epsilon_o} \right)^2 \frac{p_f^2}{(q^2 + \alpha^{-2})^2} \left(F(q^2) \right)^2$$
$$\cong \frac{4 Z^2}{(\hbar c)^4} \left(\frac{e^2}{4\pi \epsilon_o} \right)^2 \frac{(cp_f)^2}{q^4} \left(F(q^2) \right)^2$$

point charge cross-section:
most notably, falls off as q^{-4}
(units should be fm^2)

form factor squared
(dimensionless)

Check units: $[\hbar c] = [e^2 / 4\pi\epsilon_o] = \text{MeV}\cdot\text{fm}$; $[cp] = \text{MeV}$; $[q] = \text{fm}^{-1}$



$$\left[\frac{d\sigma}{d\Omega} \right] = \frac{1}{(\text{MeV}\cdot\text{fm})^4} (\text{MeV}\cdot\text{fm})^2 \frac{(\text{MeV})^2}{\text{fm}^{-4}} = \text{fm}^2$$



$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = \frac{4 Z^2}{(\hbar c)^4} \left(\frac{e^2}{4\pi \epsilon_0} \right)^2 \frac{(cp_f)^2}{q^4} \left(F(q^2) \right)^2 = Z^2 \left(\frac{d\sigma(\theta)}{d\Omega} \right)_o \left(F(q^2) \right)^2$$

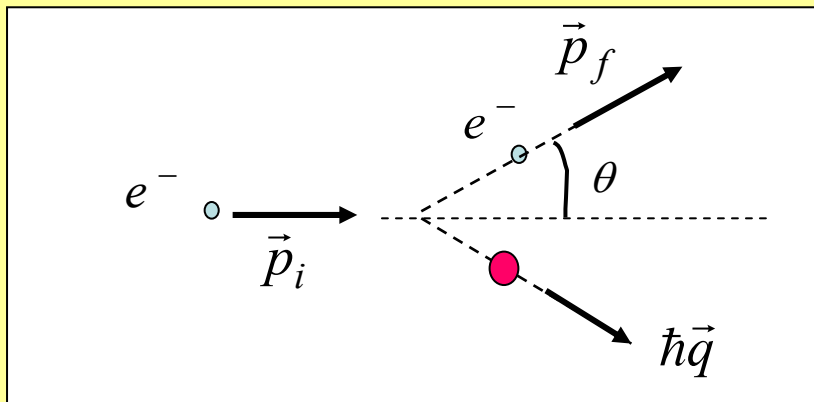
First calculate the point charge cross section for $Z = 1$:

constants:

$$\frac{4}{(\hbar c)^4} \left[\frac{e^2}{4\pi \epsilon_0} \right]^2 = \frac{4}{(197.5)^4} [1.44]^2 = 5.45 \times 10^{-9} \text{ (MeV.fm)}^{-2}$$

point charge

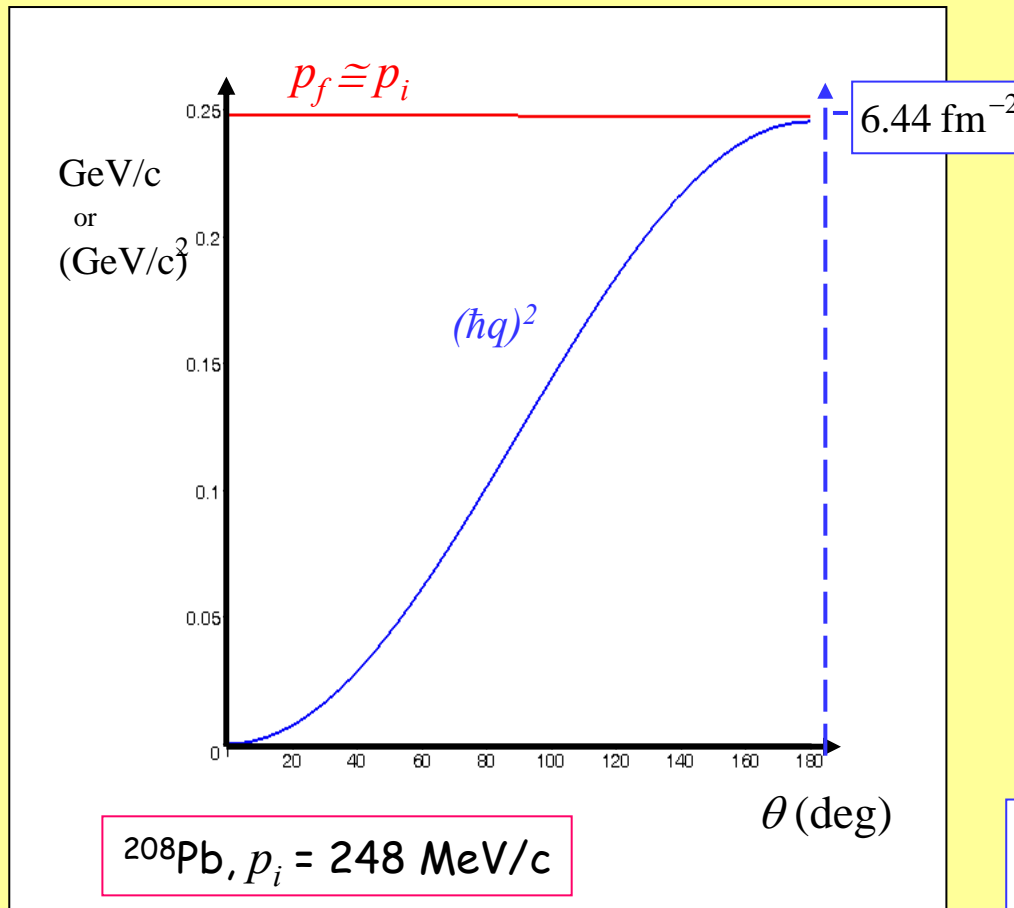
For the kinematic factors, we need our results from lecture 5 for the momenta, being of course very careful with the units:



$$p_f = \frac{p_i}{1 + \frac{p_i}{M}(1 - \cos \theta)}$$

$$(\hbar q)^2 = p_i^2 + (p_f)^2 - 2p_i p_f \cos \theta$$

First, let's consider a heavy nuclear target, so that $p_f \cong p_i$ and work out $Z^2 d\sigma/d\Omega$.
Choose an example of 248 MeV/c electrons scattering from ^{208}Pb and compare to data.



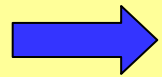
Kinematic relations become:

$$p_f \cong p_i = \text{constant}$$

$$(\hbar q)^2 \cong 2 p_i^2 (1 - \cos \theta)$$

$$\left\{ \begin{array}{l} \frac{d\sigma}{d\Omega} \sim \frac{p_f^2}{q^4} \\ 1 - \cos \theta = 2 \sin^2(\theta/2) \dots \end{array} \right.$$

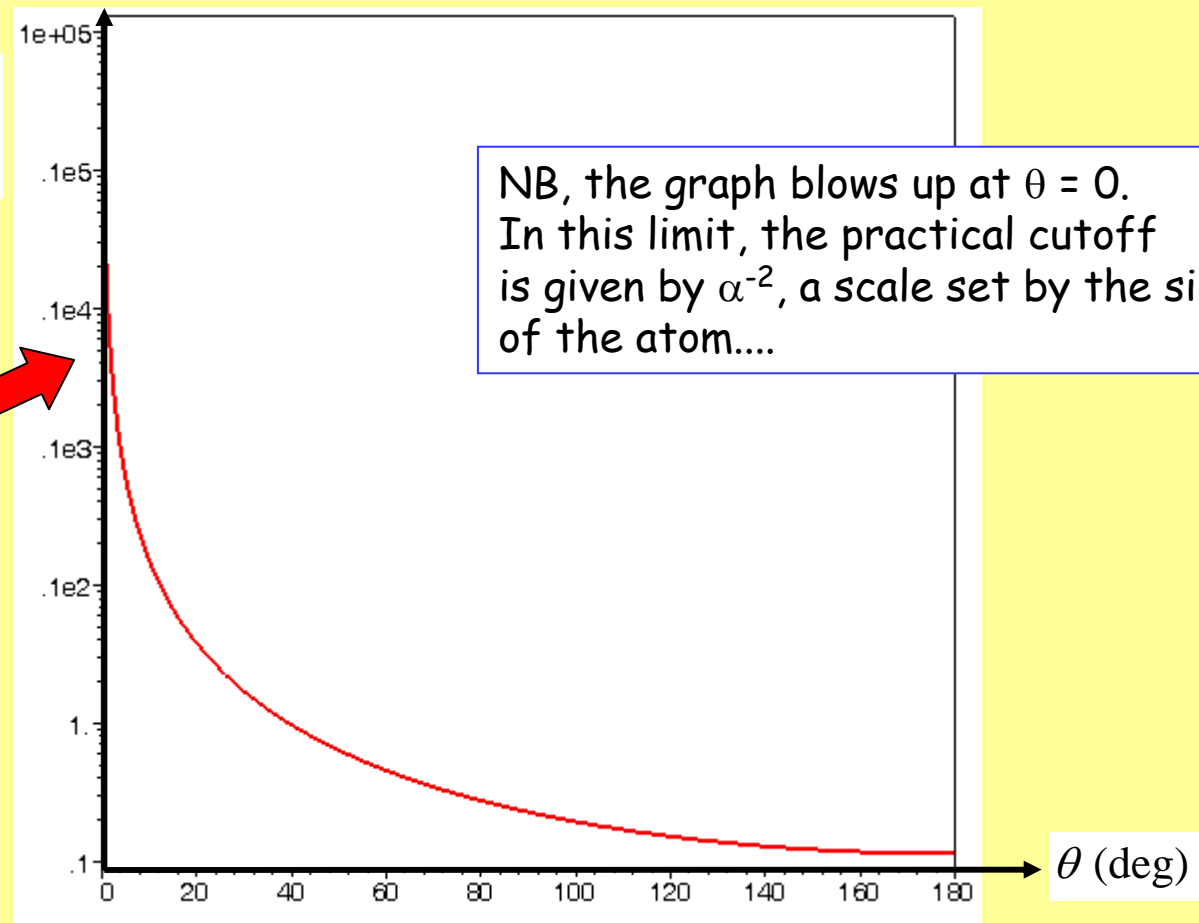
$$\left(\frac{d\sigma}{d\Omega} \right)_o = \frac{\text{const.}}{q^4} \sim \frac{1}{\sin^4(\theta/2)}$$



$$Z^2 \left(\frac{d\sigma}{d\Omega} \right)_o = \frac{\text{const.}}{q^4} = \frac{0.0567 \text{ fm}^2/\text{sr}}{\sin^4(\theta/2)}$$

$$Z^2 \left(\frac{d\sigma}{d\Omega} \right)_o \text{ fm}^2/\text{sr}$$

Log scale! The point charge cross section drops like a stone!!!



How are we doing? Compare to data:

(See F&H sec. 6.5)

11

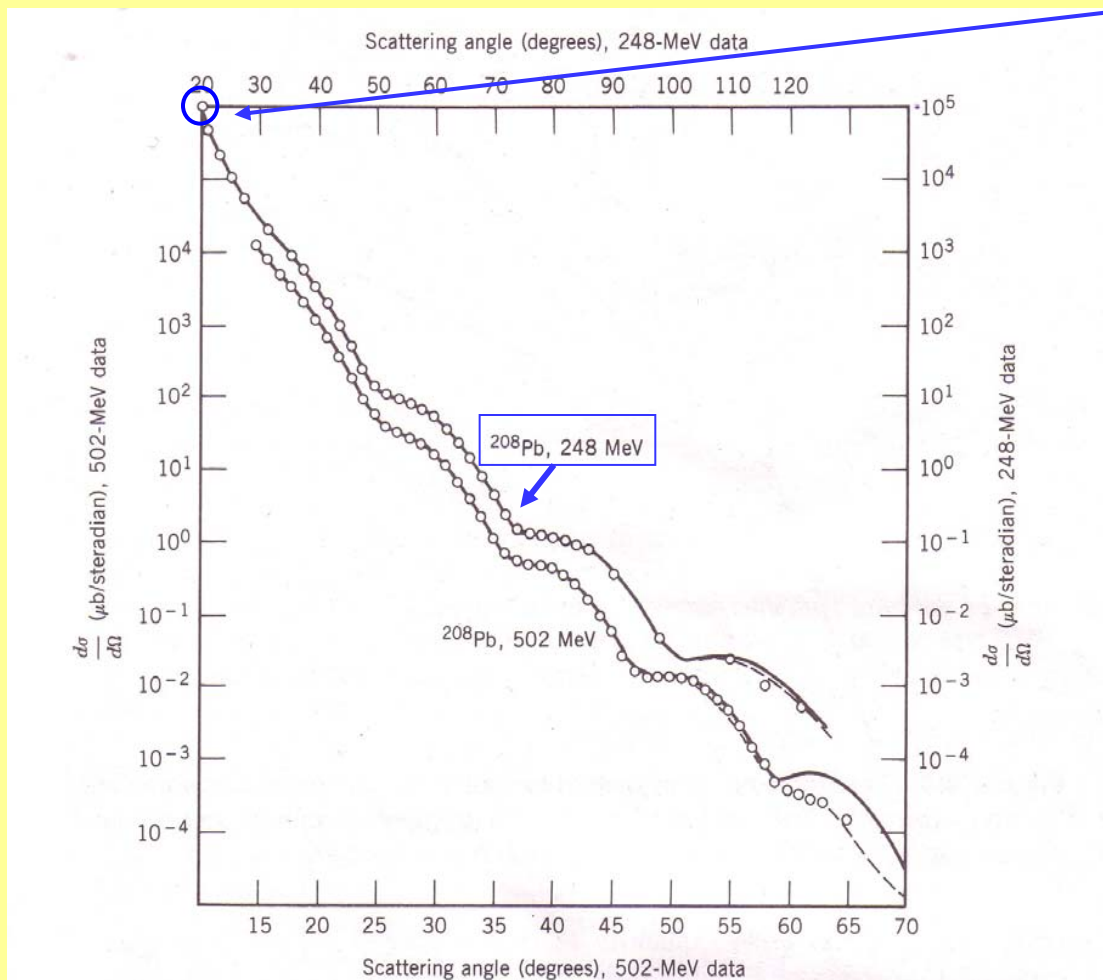


Figure 3.2 Elastic scattering of electrons from ^{208}Pb . Note the different vertical and horizontal scales for the two energies. This also shows diffractionlike behavior, but lacks sharp minima. (J. Heisenberg et al., *Phys. Rev. Lett.* **23**, 1402 (1969).)

$$\theta = 10^\circ, \\ d\sigma/d\Omega = 10^5 \mu\text{b/sr} \\ = 10 \text{ fm}^2/\text{sr} \dots$$

Point charge calculation:
 $d\sigma(10^\circ)/d\Omega = 982 \text{ fm}^2/\text{sr}$

$$\frac{d\sigma}{d\Omega} = Z^2 \left(\frac{d\sigma}{d\Omega} \right)_o (F(q^2))^2$$



$$(F(q^2))^2 \text{ at } 10^\circ = \\ 10/982 = 0.01 !!!$$

(The correction from relativistic quantum mechanics gives a $\cos^2(\frac{1}{2}\theta)$ multiplicative correction, which is trivial here.)

Also, the graphs are not smooth like $1/q^4$ -- evidence that the target has finite size!

We can predict the cross-section exactly for a pointlike target with nonrelativistic quantum mechanics.

(This approach is correct for a target particle that has charge but no magnetic moment, i.e. intrinsic angular momentum of **zero**. We can't use this for the proton without adding some refinements, so along the way we are stopping to look at the charge distributions of nuclei. Nuclei with (Z, N) both **even**, such as ^{208}Pb , have angular momentum zero, so our theory is perfect for this case!)

Recall our basic result:

$$\left(\frac{d\sigma(\theta)}{d\Omega} \right) = (\text{point charge cross-section}) \times \left(F(q^2) \right)^2$$

Where $F(q^2)$ is the Fourier transform of the target charge density:

$$F(q^2) \equiv \int e^{i\vec{q} \cdot \vec{r}} \rho(r) d^3r$$



Measuring $d\sigma/d\Omega$ and dividing by the point charge result yields a value for $F(q^2)$

form factor:

$$F(q^2) \equiv \int e^{i\vec{q} \cdot \vec{r}} \rho(r) d^3r$$

inverse Fourier transform:

$$\rho(r) \equiv \frac{1}{(2\pi)^3} \int e^{-i\vec{q} \cdot \vec{r}} F(q^2) d^3q$$

In principle, one could measure the form factor, and numerically integrate to invert the Fourier transform and find $\rho(r)$.

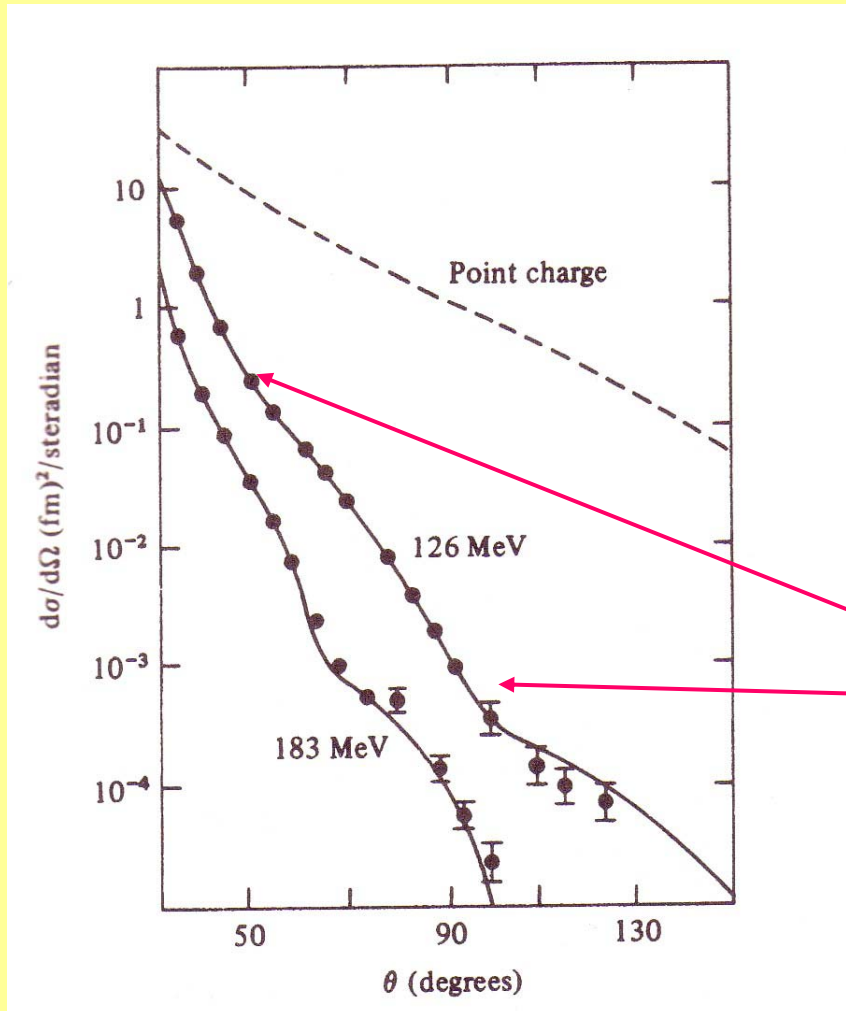
However, this doesn't work in practice, because the integral has to be done over a complete range of q from 0 to ∞ , and no experiment can ever span an infinite range of momentum transfer!

(It is bad enough trying to acquire data at large momentum transfer because the basic cross-section drops like q^{-4} → the rate of scattered particles into a detector gets too small - see lecture 4)

What to do? ...



Experimental data are fitted to a functional form for $F(q^2)$; parameters extracted from the fit are used to invert the transform and deduce $\rho(r)$



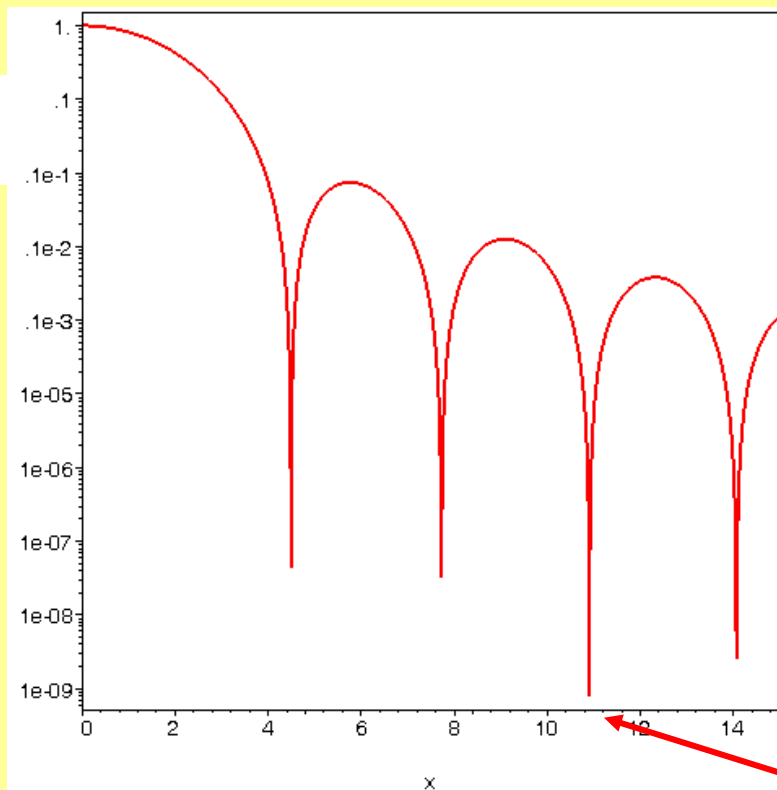
Example: elastic electron scattering from gold ($A=197$, $Z = 79$). Best fit to data is given by solid curves.

(Ref: R. Hofstadter, *Electron Scattering & Nuclear Structure*, 1963)

Discontinuities are evidence of diffraction-like behavior, characteristic of a Fourier transform, but the edges are fuzzy!

$$F(q^2) \equiv \int e^{i\vec{q} \cdot \vec{r}} \rho(r) d^3r = \int_0^{2\pi} d\phi \int_0^\infty r^2 \rho(r) dr \int_0^\pi \sin \theta e^{iqr \cos \theta} d\theta$$

$(F(q^2))^2$



Density:

$$\rho = \rho_o, \quad r < R ; \\ \rho = 0, \quad r > R$$



$$F(q^2) = \frac{3 (\sin x - x \cos x)}{x^3} \\ \{ x = qR \}$$

Important scaling property -
in dimensionless variable (qR)

Zeroes never quite get to
zero on a log scale!

In contrast, the minima are not as sharp for nuclei...

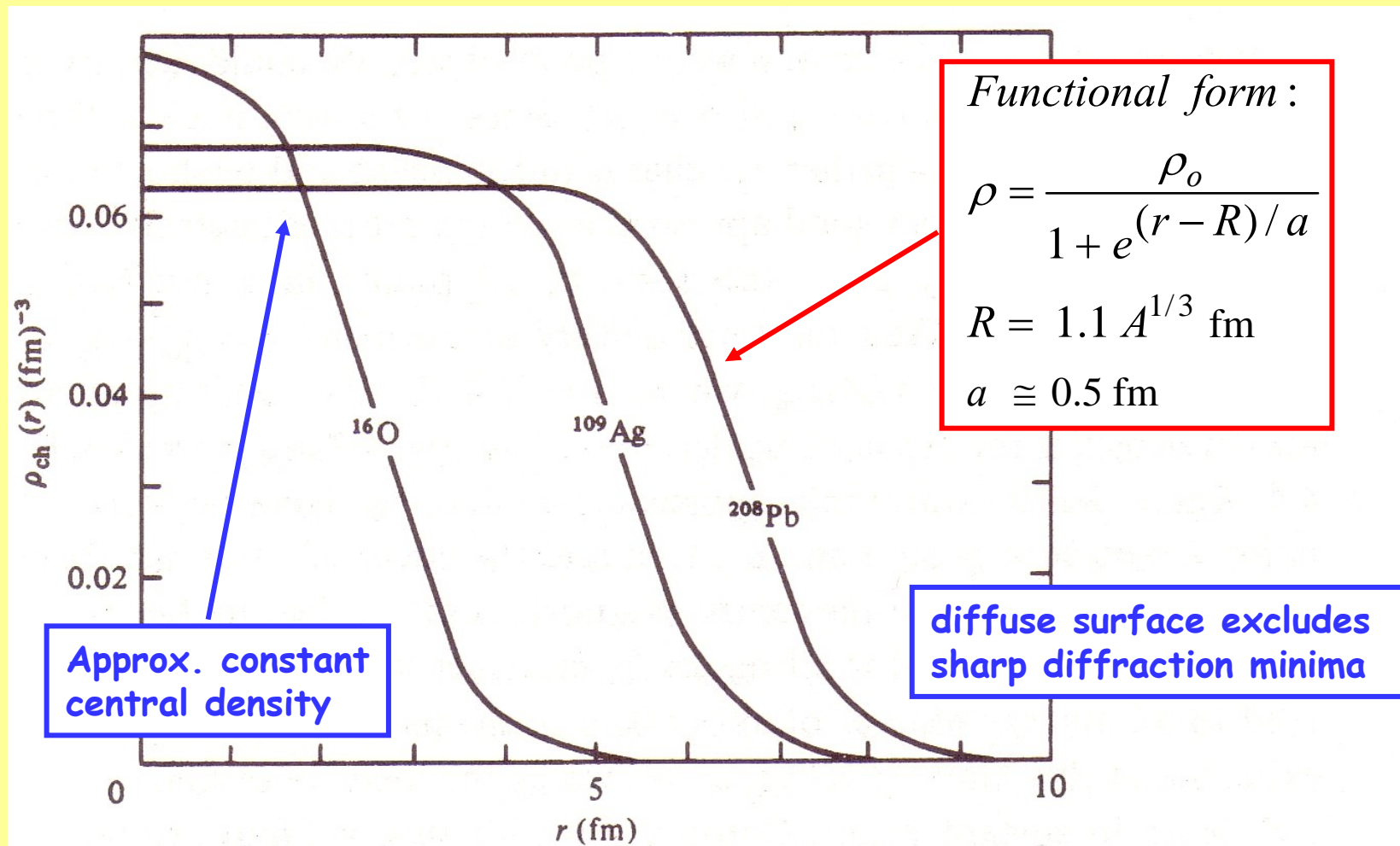
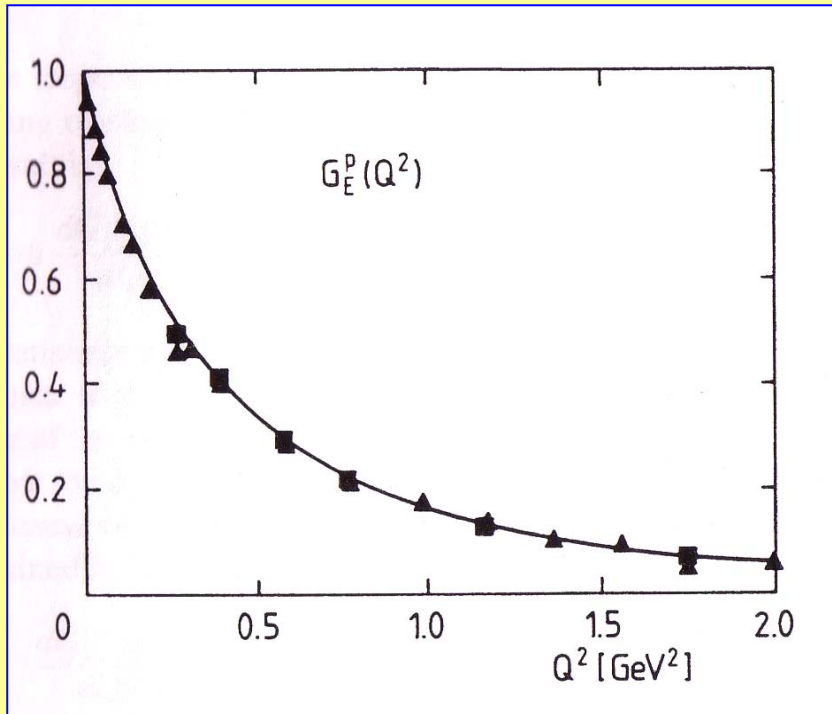


Fig. 4.3 The electric charge density of three nuclei as fitted by $\rho_{\text{ch}}(r) = \rho_{\text{ch}}^0 / [1 + \exp((r - R)/a)]$. The parameters are taken from the compilation in Barrett, R. C. & Jackson, D. F. (1977), *Nuclear Sizes and Structure*, Oxford: Clarendon Press.

Proton Form factor data:



$$G_E^p(Q^2) = \left(1 + \frac{Q^2}{0.71 \text{ GeV}^2} \right)^{-2}$$

Electric charge distribution:

$$\rho(r) = e\rho_0 \exp(-M r)$$

$$M = 4.33 \text{ fm}^{-1}$$

$$\langle r^2 \rangle^{1/2} = \frac{\sqrt{12}}{M} = 0.80 \text{ fm}$$

